NON-SEPARATING COCIRCUITS AND GRAPHICNESS IN MATROIDS

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ABSTRACT. Let M be a 3-connected binary matroid and let $Y(M)$ be the set of elements of M avoiding at least $r(M)+1$ non-separating cocircuits of M. Lemos proved that M is non-graphic if and only if $Y(M)\neq\emptyset$. We generalize this result when by establishing that $Y(M)$ is very large when M is non-graphic and M has no $M^*(K'''_{3,3})$-minor if M is regular. More precisely that $|E(M)-Y(M)|\leq 1$ in this case. We conjecture that when M is a regular matroid with an $M^*(K_{3,3})$-minor, then $r^*_M(E(M)-Y(M))\leq 2$. The proof of such conjecture is reduced to a computational verification.

1. INTRODUCTION

A cocircuit in a connected matroid is said to be non-separating if its deletion results in a connected matroid. For a 3-connected graphic matroid, note that the non-separating cocircuits correspond to the stars of the vertices in its graphic representation.

Non-separating cocircuits play an important role in the understanding of the structure of graphic matroids, as we can see in some instances that follows. Non-separating cocircuits were first studied by Tutte [18], in the cographic case, to give a characterization of the planar graphs. Tutte also proved that the non-separating cocircuits of the bond matroid of a 3-connected graph spans, over $GF(2)$, its cycle-space. Tutte's results was generalized by Bixby and Cunningham [1], as we summarize in the following theorem, which was conjectured by Edmonds:

Theorem 1.1. *Let M be a 3-connected binary matroid with at least 4 elements. Then*

(a) *the non-separating cocircuits of M span its cocircuit-space,*
(b) *each element of M is in at least two non-separating cocircuits, and*
(c) *M is graphic if and only if each element of M is in at most two non-separating cocircuits.*

Lemos, in [10] and [11], proved results of similar nature as synthesized in the next theorem:

Theorem 1.2. *Let M be a 3-connected binary matroid with at least 4 elements. Then*

(a) *for $e\in E(M)$, the non-separating cocircuits of M that avoid e span a hyperplane in the cocircuit-space of M, in particular e avoids at least $r(M)-1$ cocircuits of M. Moreover, e avoids more than $r^*(M)-1$ non-separating cocircuits of M if and only if the set of the non-separating cocircuits of M avoiding e is linearly dependent, and*
(b) *M is graphic if and only if each element of M avoids at most $r^*(M)-1$ non-separating cocircuits.*

There are another characterizations of graphicness in binary matroids using non-separating cocircuits in Kelmans [7], Lemos, Reid, and Wu [13] and Mighton [12]. Kelmans [6] gave a simple proof of Whitney's 2-isomorphism Theorem using non-separating cocircuits. Some algorithms for recognizing graphicness in binary matroids are based on concepts related to non-separating cocircuits, see Tutte [19], Cunningham [4], Mighton [12] and Wagner [16].

Theorems 1.1 and 1.2 identify two sets of obstructions for graphicness in a 3-connected binary matroid M. We define the set $X(M)$ as the set of elements of M meeting more than two non-separating cocircuits and the set $Y(M)$ of the elements of M avoiding more than $r^*(M)-1$ non-separating cocircuits. Equivalently, by Theorem 1.2, (a), we may define $Y(M)$ as the set of the elements of M avoiding all the members of a linearly dependent family of non-separating

cocircuits of M. Some sharp lower bounds for $|X(M)|$ and $|X(M)\cap Y(M)|$ are given by Lai, Lemos, Reid and Shao [8] when M is not graphic. In this work we prove that $Y(M)$ contains almost all elements of $E(M)$ if M is not graphic. Define $\widetilde{Y}(M) := E(M) - Y(M)$. The main result we establish here is:

Theorem 1.3. *Let M be a 3-connected non-graphic binary matroid. Then*
- (a) *if M is non-regular, then $|\widetilde{Y}(M)| \le 1$. Moreover, $\widetilde{Y}(M) = \emptyset$ if M has no S_8-minor or M has an $PG(3,2)$-minor.*
- (b) *if M is regular with no $M^*(K_{3,3}''')$-minor or with a $M^*(K_5)$-minor, then $\widetilde{Y}(M) = \emptyset$.*

The following conjecture generalizes this last theorem.

Conjecture 1.4. *Let M be a 3-connected non-graphic binary matroid. Then $r^*_M(\widetilde{Y}(M)) \le 2$.*

In this paper we reduce the proof of Theorem 1.3 and Conjecture 1.4. The theoretical part of the proof of conjecture is included in this paper. The computational part is being prepared. The computational part of proof of Theorem 1.3 is ready, but not properly written yet. Because of these missing pieces this paper is a preliminary report. More precisely, the theoretical part of proof of Conjecture 1.4, reduces its proof to the verification of the following:

Conjecture 1.5. *Let M be a 3-connected non-graphic regular matroid with an $M^*(K_{3,3}''')$-minor, with no $M^*(K_5)$-minor and satisfying $r^*(M) \le 9$. Then $r^*_N(\widetilde{Y}(N)) \le 2$.*

More precisely, we prove in this version of this paper:

Theorem 1.6. *Conjectures 1.4 and 1.5 are equivalent.*

Conjecture 1.4 yields the following generalization of Lemos' graphicness criterion:

Conjecture 1.7. *Let M be a 3-connected binary matroid and $S \subseteq E(M)$ satisfy $r^*_M(S) \ge 3$. Then the following assertions are equivalent:*
- (a) *M is graphic;*
- (b) *Each element of S avoids at most $r^*(M) - 1$ non-separating cocircuits of M; and*
- (c) *Each element of S avoids no linearly dependent set of non-separating cocircuits of M.*

2. Some results in critical 3-connectivity

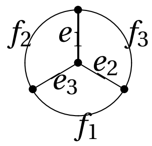
Let M and N be 3-connected matroids. We say that an element $e \in E(M)$ is *vertically N-removable* in M if $co(M\backslash e)$ is a 3-connected matroid with an N-minor.

We can summarize the dual versions of Lemma 3.4 and Theorem 3.1 of [17] and Theorem 1.3 of [3] in the following theorem:

Theorem 2.1. *Let M and N be 3-connected matroids. For $k \in \{1,2,3\}$, if M has an N-minor and $r^*(M) - r^*(N) \ge k$, then there is a k-coindependent set of M whose elements are vertically N-removable in M.*

If M and N are 3-connected matroids, we say that a set $X \subseteq E(M)$ is *N-removable* if $M\backslash X$ is a 3-connected matroid with an N-minor. We also say that an element $e \in E(M)$ is *N-removable* if $\{e\}$ is N-removable. Now, we define a special structure that will be largely used along this article. When M is binary, we say that a list of distinct elements $e_1, e_2, e_3, f_1, f_2, f_3$ of M is an *N-pyramid* with *top* $T^* := \{e_1, e_2, e_3\}$ and *base* $T := \{f_1, f_2, f_3\}$ provided:
- (a) T^* is an N-removable triad of M,
- (b) T is a triangle of M,
- (c) for $\{i, j, k\} = \{1,2,3\}$, $\{e_i, e_j, f_k\}$ is a triangle of M, and
- (d) the elements f_1, f_2 and f_3 are N-removable in M.

In other words, such a list is an N-pyramid with top T^* when the restriction of M to its elements is isomorphic $M(K_4)$ as illustrated in the figure below, where $\{e_1, e_2, e_3\}$ is an N-removable triad of M and f_1, f_2 and f_3 are also N-removable in M.



Theorem 2.1 does not hold for $k > 3$, but, at the binary case, it can be extended by the following theorem, that is the dual version of Theorem 1.4, from [3], written in terms of N-pyramids.

Theorem 2.2. *Let M be a 3-connected binary matroid with a 3-connected minor N such that $r^*(M) - r^*(N) \geq 5$. Then*

(a) *M has a 4-coindependent set whose elements are vertically N-removable, or*
(b) *M has an N-pyramid.*

3. Lifting non-separating cocircuits from minors

We define $\mathscr{R}^*_A(M)$ as the set of non-separating cocircuits of M avoiding A. We may write $R^*_e(M)$ instead of $R^*_{\{e\}}(M)$. We write $dep_A(M) := |\mathscr{R}^*_A(M)| - dim(\mathscr{R}^*_A(M))$, where $dim(\mathscr{R}^*_A(M))$ is the dimension of the space spanned by $\mathscr{R}^*_A(M)$ in the cocircuit space of M. We simplify the notation $dep_{\{e\}}(M)$ by $dep_e(M)$.

Observe that an element e of a 3-connected binary matroid M is in $Y(M)$ if and only if $dep_e(M) > 0$. The next result is Lemma 3.1 from [11].

Lemma 3.1. *Suppose that e is an element of a 3-connected binary matroid M such that the cosimplification of $M\backslash e$ is 3-connected. If $r^*(M) \geq 4$, then it is possible to choose the ground set of $co(M\backslash e)$ so that, for each $A \subseteq E(co(M\backslash e))$,*

$$dep_A(M) \geq dep_{A'}(M) \geq dep_A(co(M\backslash e)), \tag{1}$$

where A' is the minimal subset of $E(M)$ satisfying $A \subseteq A'$ and, for each triad T^* of M that meets both e and A, $T^* - e \subseteq A'$.

Next we establish:

Lemma 3.2. *Suppose that M is a 3-connected binary matroid satisfying $r^*(M) \geq 4$. If e is an element of M such that $co(M\backslash e)$ is 3-connected, then we may choose the ground set of $co(M\backslash e)$ so that:*

$$\begin{aligned} &Y(co(M\backslash e)) \subseteq Y(M);\ \textit{and} \\ &\{f \in E(M) :\ \textit{there is}\ g \in Y(co(M\backslash e))\ \textit{with}\ \{f,g\} \in \mathscr{C}^*(M\backslash e)\} \subseteq Y(M). \end{aligned} \tag{2}$$

Proof. Choose the ground set of $co(M\backslash e)$ satisfying condition (1) of Lemma 3.1. Recall that $f \in Y(M)$ if and only if $dep_f(M) \geq 1$. If $f \in Y(co(M\backslash e))$, then, by (1), $dep_f(M) \geq dep_f(co(M\backslash e)) \geq 1$ and, therefore $f \in Y(M)$. Now, suppose that $\{e, f, g\}$ is a triad of M and that g in $Y(co(M\backslash e))$. By (1), $dep_{\{f,g\}}(M) \geq dep_g(M) > 0$. Thus $\mathscr{R}^*_{\{f,g\}}(M)$ is linearly dependent and, therefore, so is $\mathscr{R}^*_f(M)$, since it contains $\mathscr{R}^*_{\{f,g\}}(M)$. Thus $f \in Y(M)$. $\square$

Corolary 3.3. *Suppose that M is 3-connected binary matroid satisfying $r^*(M) \geq 4$. If e is an element of M such that $co(M\backslash e)$ is 3-connected, then we may choose the ground set of $co(M\backslash e)$ so that:*

$$\widetilde{Y}(M) \subseteq cl^*_M(\widetilde{Y}(co[M\backslash e]) \cup e). \tag{3}$$

Lemma 3.4. *Let M be a 3-connected binary matroid with an N-minor, satisfying $r^*(M) \geq 4$. Suppose that $e_1, e_2, e_3, f_1, f_2, f_3$ is an N-pyramid of M, having top T^* and base T. Denote $F := T \cup T^*$.*

(a) *If $D^* \in \mathscr{R}^*(M)$, then either $D^* = T^*$, $D^* \cap F = \emptyset$, or $D^* \cap F = \{e_i, f_j, f_k\}$, for some distinct elements $i, j, k \in \{1,2,3\}$.*

(b) *If $C^* \in \mathscr{R}^*(M\backslash T^*)$, then there is a unique $D^* \in \mathscr{R}^*(M)$ such that $C^* \subseteq D^* \subsetneq D^* \cup T^*$. Moreover, either*
  (b1) *$C^* \cap T = \emptyset$ and $D^* = C^*$, or*
  (b2) *there are distinct elements i, j, k of $\{1,2,3\}$ such that $C^* \cap F = \{f_i, f_j\}$ and $D^* = C^* \cup e_k$.*

(c) *$Y(M\backslash T^*) \subseteq Y(M)$. Moreover, if $f_i \in Y(M\backslash T^*)$, then $T_i \subseteq Y(M\backslash T^*)$ and $f \notin cl^*_M(\widetilde{Y}(M))$.*

(d) *If $r^*_{M\backslash T^*}(\widetilde{Y}(M\backslash T^*)) \leq 2$ and, for each $i = \{1,2,3\}$, $r^*_{M\backslash f_i}(\widetilde{Y}(M\backslash f_i)) \leq 2$, then $r^*_M(\widetilde{Y}(M)) \leq 2$.*

Proof. In this proof we set, for $\{i, j, k\} = \{1,2,3\}$, $T_k := \{e_i, e_j, f_k\}$.

Let us prove (a). Suppose that $D^* \in \mathscr{R}^*(M)$, $D^* \neq T^*$ and $D^* \cap F \neq \emptyset$. As D^* intersects F, it follows that D^* intersect T_k for some k. By orthogonality with T_k, D^* intersects T^*. Hence $|D^* \cap T^*| = 1$, since $M\backslash D^*$ is connected; say $e_1 \in D^*$. By orthogonality with T_2 and T_3, it follows that $\{e_1, f_2, f_3\} \subseteq D^*$. As M is binary, none of T, T_2 or T_3 is contained in D^*. It yields that $D^* \cap F = \{e_1, f_2, f_3\}$. We proved (a).

Let us prove (b). First, we examine the case that $C^* \cap T = \emptyset$. In this case, it is straight to check that $M\backslash C^*$ is connected. It is just left to show that C^* is a cocircuit of M. Consider a cocircuit D^* of M such that $C^* \subseteq D^* \subsetneq C^* \cup T^*$, say $e_1 \notin D^*$. As $C^* \cap T = \emptyset$, then $f_2, f_3 \notin D^*$. By orthogonality with T_2 and T_3, $T^* \cap D^* = \emptyset$. Thus $D^* = C^*$ and C^* is a cocircuit of M. Moreover, in this case, (b1) holds. So we may assume that $C^* \cap T \neq \emptyset$. By orthogonality with T, we may suppose that $C^* \cap T = \{f_1, f_2\}$. Let D^*_0 be a cocircuit of M such that $C^* \subseteq D^*_0 \subsetneq C^* \cup T^*$. By orthogonality with T_1 and T_2, either $D^*_0 = C^* \cup e_3$ or $D^*_0 = C^* \cup \{e_1, e_2\}$. Note that $C^* \cup \{e_1, e_2\} = (C^* \cup e_3) \Delta T^*$. Thus both $C^* \cup e_3$ and $C^* \cup \{e_1, e_2\}$ are cocircuits of M. But it is easy to check that that $M\backslash(C^* \cup e_3)$ is connected and $M\backslash(C^* \cup \{e_1, e_2\})$ is disconnected. Define $D^* := C^* \cup e_3$ to conclude (b) and (b2).

To prove (c), let $e \in Y(M\backslash T^*)$. As $\mathscr{R}^*_e(M)$ is linearly dependent, there are distinct non-separating cocircuits $C^*_1, \dots, C^*_n$ of $M\backslash T^*$ avoiding e such that:

$$C^*_1 \Delta \dots \Delta C^*_n = \emptyset. \tag{4}$$

For each $l = 1, \dots, n$, define D^*_l as the non-separating cocircuit of M such that $C^*_l \subseteq D^*_l \subseteq C^*_l \cup T^*$, as described in (b). Consider, for $\{i, j, k\} = \{1,2,3\}$, the following subsets of $\{1, \dots, n\}$:

$$\begin{aligned} B_i &:= \{l : f_i \in C^*_l\}, \text{ and} \\ A_{ij} &:= \{l : f_i, f_j \in C^*_l\} = \{l : e_k \in D^*_l\} \text{ (this equality holds by (a)).} \end{aligned}$$

By (4), each B_i has even cardinality. By (a), B_i is equal to the disjoint union of A_{ij} and A_{ik}. Thus $|A_{12}|, |A_{13}|$ and $|A_{23}|$ are congruent modulo 2. Hence $D^*_1 \Delta \dots \Delta D^*_n$ is equal to $\emptyset$ or T^*. Therefore $D^*_1, \dots, D^*_n$ or $D^*_1, \dots, D^*_n, T^*$ is a list of linearly dependent non-separating cocircuits of M avoiding e. This proves the first part of (c).

For the second part of (c), say that $f_1 \in Y(M\backslash T^*)$. Note that, as above, for $e = f_1$ we have $A_{12} = A_{13} = \emptyset$. Thus $|A_{23}|$ is even and, therefore, $D^*_1 \Delta \dots \Delta D^*_n = \emptyset$. In particular this implies that $D^*_1, \dots, D^*_n$ is a list of linearly dependent cocircuits of M avoiding T_1. To finish, note that, in this case, $\widetilde{Y}(M)$ is contained in the cohyperplane $E(M) - T_1$ of M and, therefore, $f_1 \notin cl^*_M(\widetilde{Y}(M))$.

Now, we prove (d). Suppose, for a contradiction, that $r^*_M(\widetilde{Y}(M)) \geq 3$. As $M\backslash T^*$ is 3-connected and binary and T is a triangle of $M\backslash T^*$, and, by hypothesis, $r^*_{M\backslash T^*}(\widetilde{Y}(M\backslash T^*)) \leq 2$, it follows that $T \nsubseteq \widetilde{Y}(M\backslash T^*)$, say $f_1 \notin \widetilde{Y}(M\backslash T^*)$. Thus $f_1 \in Y(M\backslash T^*)$. By (c), $f_1 \notin \widetilde{Y}(M)$. As, $M\backslash f_1$ is 3-connected, by Corollary 3.3 for $e = f_1$, we have that $\widetilde{Y}(M) \subseteq cl^*_M(\widetilde{Y}(M\backslash f_1) \cup f_1)$. But $r^*_M(\widetilde{Y}(M\backslash f_1) \cup f_1) \leq 3$, because $r^*_{M\backslash f_1}(\widetilde{Y}(M\backslash f_1)) \leq 2$. Thus, as $r^*_M(\widetilde{Y}(M)) \geq 3$, $\widetilde{Y}(M)$ spans f_1 in M^*. A contradiction to the second part of (c). This proves (d), and therefore, the lemma. $\square$

The next lemma is a straight consequence of the submodularity of the rank function of a matroid.

Lemma 3.5. *Let M be a matroid, $X_1,\dots,X_m \subseteq E(M)$ and $n := \max_i r_M(X_i)$. Suppose that $r_M(X_1 \cup \dots \cup X_m) \geq n+1$. Then $r_M(X_1 \cap \dots \cap X_m) \leq n-1$.*

Lemma 3.6. *Let $l \in \{0,1,2\}$ and M and N be 3-connected binary matroids. Suppose that M has an N-minor and $r^*(M) \geq 4$. If M has a $(l+2)$-coindependent set I^*, such that for each $e \in I^*$, e is vertically N-removable and $r^*_{co(M\backslash e)}[\widetilde{Y}(co(M\backslash e))] \leq l$, then $r^*_M(\widetilde{Y}(M)) \leq l$.*

Proof. Let $I^* := \{e_1,\dots,e_{(l+2)}\}$. For $i = 1,\dots,l+2$, we choose the ground set of $M_i := co(M\backslash e_i)$ satisfying equation (3), in Corollary 3.3. So we have that, for each i, $\widetilde{Y}(M) \subseteq X_i := cl^*_M(\widetilde{Y}(M_i) \cup e_i)$. Our hypothesis implies that, for each i, $r^*_M(X_i) \leq l+1$. Define $X := X_1 \cup \dots \cup X_{(l+2)}$. As $I \subseteq X$ and $r^*_M(X) \geq l+2$ then, by the dual version of Lemma 3.5 for $n = l+1$, $r^*_M(X_1 \cap \dots \cap X_m) \leq l$. Thus $r^*(\widetilde{Y}(M)) \leq l$, since $\widetilde{Y}(M) \subseteq X_1 \cap \dots \cap X_{l+2}$. □

Lemma 3.7. *Let M be a 3-connected binary matroid with a 3-connected minor N with $r^*(M) \geq 4$ and let $l \in \{0,1,2\}$. If $r^*(M) - r^*(N) \geq 2 + l + \left\lfloor \frac{l}{2} \right\rfloor$ and $r_M(\widetilde{Y}(M)) \geq l+1$, then*

(a) *M has a vertically N-removable element e such that $\widetilde{Y}(co[M\backslash e])$ has rank at least $l+1$ in $co(M\backslash e)$, or*
(b) *$l = 2$ and M has an N-pyramid e_1,e_2,e_3,f_1,f_2,f_3 such that there is $K \in \{M\backslash\{e_1,e_2,e_3\}, M\backslash f_1, M\backslash f_2, M\backslash f_3\}$ satisfying $r_K(\widetilde{Y}(K)) \geq 3$.*

Proof. By Theorems 2.1 and 2.2, one of the two statements holds:

(i) M has an $(l+2)$-coindependent set $I := \{e_1,\dots,e_{(l+2)}\}$ whose elements are vertically N-removable; or
(ii) $l = 2$ and M has an N-pyramid e_1,e_2,e_3,f_1,f_2,f_3, with top T^*.

By lemma 3.6, (i) implies (a). By lemma 3.4, (d), (ii) implies (b). The lemma is proved. □

4. Some initial cases

The proof of the next lemma is just a routine check.

Lemma 4.1. *If $M \in \{F_7, F_7^*, M^*(K_5), R_{10}\}$ then $Y(M) = E(M)$.*

Consider the partition of the vertices of $K_{3,3,}$ into two stable sets V_1 and V_2. For $0 \leq j \leq i \leq 3$, we define $K_{3,3}^{(i,j)}$ by a simple graph obtained from $K_{3,3}$ by the addiction of i edges joining vertices of V_1 and j edges joining vertices of V_2. We also consider the already established notations $K'_{3,3} := K_{3,3}^{(1,0)}$, $K''_{3,3} := K_{3,3}^{(2,0)}$ and $K'''_{3,3} := K_{3,3}^{(3,0)}$.

We define a circuit C in a connected matroid M to be *non-separating* if C is a non-separating cocircuit of M^*, that is, if M/C is connected. We also say that a circuit C of a 3-connected graph G is *non-separating*, if $E(C)$ is a non-separating circuit of $M(G)$.

Lemma 4.2. *Suppose that M is a simple cographic matroid with an $M^*(K_{3,3})$-minor such that $r^*(M) = 5$. Then*

(a) *If $M \not\cong M^*(K'''_{3,3})$, then $Y(M) = E(M)$.*
(b) *If $M = M^*(K'''_{3,3})$, then $\widetilde{Y}(M)$ is a triad of M and $Y(M) = E(K_{3,3})$.*
(c) *M has a $M^*(K_5)$-minor if and only if $M \cong M^*(K_{3,3}^{(i,j)})$ for some $i,j \in \{1,2,3\}$.*

Proof. First note that $M \cong M^*(K_{3,3}^{(i,j)})$ for some $0 \leq j \leq i \leq 3$.

Item (c) follows from the facts that $K_{3,3}'''$ has no K_5-minor and that $K_{3,3}^{(1,1)}$ has a K_5-minor. Indeed, $|si(K_{3,3}'''/e)| < 10$ for all $e \in E(K_{3,3}''')$ and $si(K_{3,3}^{(1,1)}/f) \cong K_5$ where f is the edge joining the two degree-3 vertices of $K_{3,3}^{(1,1)}$.

Let us verify (a). First suppose that $M \cong M^*(G)$ for some connected graph extending $K_{3,3}^{(1,1)}$. Consider the edge f such that $co(M\backslash f) \cong M^*(K_5)$, as before. By Corollary 3.3 and Lemma 4.1, it follows that $\widetilde{Y}(M) \subseteq \{f\}$. It is just left to show that f avoids a linearly dependent set of non-separating circuits in G to finish this case. In fact, note that the triangles of G that does not contain the end-vertices of f constitute such a set. Now, we verify (a) for the remaining graphs:

(1) $K_{3,3}$: note that each 4-circuit of $K_{3,3}$ is non-separating. Let $g \in E(K_{3,3})$ and v an end-vertex of g. Note that the set of the 4-cocircuits of $K_{3,3}$ avoiding v is linearly dependent. Thus $g \in Y(M^*(K_{3,3}))$, and so $Y(M^*(K_{3,3})) = E(M^*(K_{3,3}))$.

(2) $K_{3,3}'$, : this graph has orbits , and of the automorphism group of its bond matroid. The set of representatives of the first two orbits avoid the list , , of linearly dependent non-separating circuits, while the element of the third orbit avoids , , and .

(3) $K_{3,3}''$, : analogously, we have orbits , and . The linearly dependent non-separating circuits , and avoid the two first orbits, while the representative of the third orbit avoids , , , and .

It is remaining to prove (b). Note that in $K_{3,3}'''$, , the edge avoids exactly the following non-separating circuits: , , , , and , that constitute a linearly independent set. So, the orbit is contained in $\widetilde{Y}(M^*(K_{3,3}'''))$. But the representative of the other orbit avoids the linearly dependent circuits , , and . This finishes the proof of (b) and of the lemma. □

Lemma 4.3. *If M is a 3-connected regular matroid with an $M^*(K_5)$-minor and $r^*(M) \le 5$, then M is cographic and $Y(M) = E(M)$.*

Proof. First let us verify that M is cographic. As $r(M) \le 5$, M has no R_{12}-minor. If M has an R_{10}-minor, as R_{10} is a splitter for the class of the regular matroids, then $M \cong R_{10}$; a contradiction. Thus M is cographic. If $r(M) = 4$, then $M \cong M^*(K_5)$ and the lemma follows. So, we may assume that $r(M) = 5$. As $M^*(K_5)$ is a splitter for the class of the cographic matroids with no $M^*(K_{3,3})$-minor, then M is isomorphic to the bond matroid of a graph with 6 vertices extending $K_{3,3}$. The lemma follows from items (a) and (c) of Lemma 4.2. □

The next lemma has a computer assisted proof that will be approached in Section 8.

Lemma 4.4. *Let M be a 3-connected binary matroid and $e \in E(M)$.*

(a) *If $co(M\backslash e) \cong S_8$, then $|\widetilde{Y}(M)| \le 1$.*
(b) *If $r^*(M) = 4$ and $M \ncong S_8$, then $Y(M) = E(M)$. Moreover $|\widetilde{Y}(S_8)| = 1$.*
(c) *If $co(M\backslash e)$ is isomorphic to $M^*(K_{3,3}^{(i,0)})$ for some $i \in \{0,1,2\}$, then $Y(M) = E(M)$.*
(d) *If M has an element e such that $co(M\backslash e) \cong PG(3,2)^*$ then $E(M) = Y(M)$.*

5. Proof of the main theorem

The statement of Theorem 1.3 just summarizes the lemmas proved in this section.

Lemma 5.1. *If M is a regular matroid with a $M^*(K_5)$-minor, then $E(M) = Y(M)$.*

Proof. Suppose the M is a minimal counter-example to the lemma. By Lemma 4.3, $r^*(M) \ge 6$. So, by Lemma 3.7, for $l = 0$, M has a vertically $M^*(K_5)$-removable element e such that $\widetilde{Y}(co(M\backslash e))$ is non-empty; a contradiction to the minimality of M. □

Lemma 5.2. *If M is a non-graphic regular matroid with no $M^*(K'''_{3,3})$-minor then $E(M) = Y(M)$.*

Proof. Suppose that M is a minimal counter-example for the lemma. By lemma 5.1, M has no $M^*(K_5)$-minor. Thus, since M is not graphic, M has a $M^*(K_{3,3})$-minor.

If $r^*(M) \geq 7$, then, by Lemma 3.7 for $l = 0$, it follows that M has a vertically $M^*(K_{3,3})$-removable element e such that $\widetilde{Y}(co(M\backslash e)) \neq \emptyset$. But it contradicts the minimality of M. Hence $r^*(M) \leq 6$.

Note that M is not isomorphic to R_{10} nor have an R_{10}-minor, since R_{10} is a splitter for the class of the regular matroids. If $r^*(M) = 6$, M has a vertically $M^*(K_{3,3})$-removable element e. As $co(M\backslash e)$ has no minor isomorphic to R_{10} or R_{12}, it follows that $co(M\backslash e)$ is cographic. In particular, M is a corank-5 cographic matroid extending $M^*(K_{3,3})$. By Lemma 4.2(c), $co(M\backslash e)$ is isomorphic to $M^*(K^{(i,0)}_{3,3})$ for some $i \in \{0,1,2\}$. In this case the result follows from Lemma 4.4(c). Thus $r^*(M) = 5$. As before, M has no R_{10} or R_{12}-minor and M is cographic. Now the result follows from Lemma 4.2, (a). □

Lemma 5.3. *Suppose that M is a 3-connected non-regular binary matroid. Then $|\widetilde{Y}(M)| \leq 1$. Moreover, if M has no S_8-minor, then $Y(M) = E(M)$.*

Proof. Note that F_7 satisfies the lemma. But F_7 and F_7^* are the unique excluded minors for regularity in the class of the binary matroids. Moreover F_7^* is a splitter for the class of binary matroids with no F_7-minor. Thus we may assume that M has a F_7^*-minor.

First suppose that M is a minimal counter-example for the first part of the lemma. By Lemma 4.4(b), $r^*(M) \geq 5$. If $r^*(M) = 5$ then, by Lemma 3.7 (for $l = 0$), M has an element e such that $\widetilde{Y}(co(M\backslash e)) \neq \emptyset$. By Lemma 4.4(b) again, $co(M\backslash e) \cong S_8$. But it contradicts Lemma 4.4(a). Thus $r^*(M) \geq 6$. It follows by Lemma 3.7 (for $l = 1$), that M has a vertically F_7^*-removable element e such that $|\widetilde{Y}(co(M\backslash e)| \geq 2$. Thus $co(M\backslash e)$ contradicts the minimality of M.

Now suppose that M is a minimal counter-example for the second part of the lemma. If $r^*(M) \geq 5$, then, by Lemma 3.7, for $l = 0$, M has a vertically F_7^*-removable element e such that $\widetilde{Y}(co(M\backslash e)) \neq \emptyset$, a contradiction to the minimality of M. Thus $r^*(M) = 4$. Now, the result follows from Lemma 4.4(b). □

Lemma 5.4. *If M is a binary 3-connected matroid with a $PG(3,2)^*$-minor, then $Y(M) = E(M)$.*

Proof. Let M be a minimal counter-example for the lemma. If $r^*(M) \geq 6$, then, by Lemma 3.6 for $l = 0$, we have a contradiction to the minimality M. So, $r^*(M) \leq 5$, but, by Lemma 4.4(b), $r^*(M) \geq 5$. Thus $r^*(M) = 5$. By Theorem 2.1, M has a vertically $PG(3,2)^*$-removable element e. Thus $co(M\backslash e) \cong PG(3,2)^*$, since $PG(3,2)^*$ is a maximal rank-4 binary matroid. But it contradicts Lemma 4.4(v). □

Proof of Theorem 1.6: It is clear that Conjecture 1.5 is a particular case of Conjecture 1.4. In other hand if we consider a minimal counter-example M for the converse, by Lemma 3.7 (for $l = 2$), analogously to the preceding proofs, M has a minor that contradicts the minimality of M. □

6. EXTREMAL CASES FOR THE MAIN THEOREM

Here we denote by Z_r the *binary rank-r spike*: a matroid represented by a binary $(2r+1) \times r$ matrix in the form $[I_r|\bar{I}_r|\vec{1}]$, where $\bar{I}_r$ is I_r with the symbols interchanged and $\vec{1}$ is a column full of ones. We use the respective labels $a_1, \dots, a_r, b_1, \dots, b_r, c$ in this representation. We also define, for $n \geq 4$, $S_{2n} := Z_n\backslash b_n$.

Proposition 6.1. *For $n \geq 4$, S_{2n} attains the bound $|\widetilde{Y}(S_{2n})| = 1$ in Theorem 1.3.*

Proof. By theorem 1.3(i), we have that $\widetilde{Y}(S_{2n}) \leq 2$. So it is enough to verify that $\widetilde{Y}(S_{2n}) \neq \emptyset$. As S_{2n} is self-dual we prove the proposition by showing that a_n avoids at most $n-1$ non-separating circuits of S_{2n}. Note that the spanning circuits of S_{2n} are not non-separating. It is not hard to verify that the non-spanning circuits of S_{2n} avoiding a_n are those in the form $\{c, a_i, b_i\}$, for some $1 \leq i < n-1$, or in the form $\{a_i, b_i, a_j, b_j\}$, for some $1 \leq i < j < n-1$ (the reader may also see [15], page 662). But c is a loop of the matroids in the form $S_{2n}/\{a_i, b_i, a_j, b_j\}$, which are, therefore, disconnected. Thus a_n avoids at most $n-1$ non-separating circuits. □

Let $n \geq 3$ and let V_1 and V_2 be the members of a partition of the vertex set of $K_{3,n}$ into two stable sets, where $|V_1| = 3$. Define $K'''_{3,n}$ as the graph obtained from $K_{3,n}$ by adding an edge joining each pair of vertices in V_1. Note that the unique non-separating cocircuits of $M^*(K'''_{3,n})$ are the triangles of $K'''_{3,n}$ meeting both V_1 and V_2. So, $\widetilde{Y}(M^*(K_{3,n}))$ is the triad $E(K'''_{3,n}) - E(K_{3,n})$. Thus we have an infinite set of matroids attaining the bound $r^*_M(\widetilde{Y}(M)) = 2$ in Theorem 1.3 (b).

7. Complementary matroids in relation to projective geometries and a handmade classification of the rank-4 3-connected binary matroids

From the uniqueness of representability of binary and ternary matroids, and from [15, 6.3.15], we may conclude that:

Lemma 7.1. *Let $q \in \{2,3\}$, $s \geq 2$, and $X, Y \subseteq E(PG(s,q))$. Suppose that there is a matroid isomorphism $\varphi : PG(s,q)|X \to PG(s,q)|Y$. Then, there is an automorphism Φ of $PG(s,q)$ that extends φ and which restriction to $E(PG(s,q)) - X$ is a matroid isomorphism between $PG(s,q)\backslash X$ and $PG(s,q)\backslash Y$.*

If, for $q \in \{2,3\}$, M is a rank-r simple matroid representable over $GF(q)$ we have, for $s \geq r-1$, well defined up to isomorphisms, the *complementary of M in relation to $PG(s,q)$* as the matroid $PG(s,r)\backslash M := PG(s,q)\backslash X$, where $X \subseteq E(PG(s,q))$ is a set that satisfies $M \cong PG(s,q)|X$. Lemma 7.1 implies:

Corolary 7.2. *Let $q \in \{2,3\}$, let M and N be simple rank-r matroids representable over $GF(q)$ and let $s \geq max\{r(M), r(N)\} - 1$. Then*

(a) *$M \cong N$ if, and only if, $PG(s,q)\backslash M \cong PG(s,q)\backslash N$; and:*
(b) *M is isomorphic to a minor of N if, and only if, $PG(s,q)\backslash N$ is isomorphic to a minor of $PG(s,q)\backslash M$.*

Theorem 7.3. *Let $\mathbb{P} := PG(3,2)$. Up to isomorphisms, all the rank-4 binary 3-connected matroids are:*

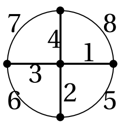
(i) *F_7^*, S_8, $AG(3,2)$ and $M(\mathcal{W}_4)$, up to 8 elements;*
(ii) *$Z_4 \cong \mathbb{P}\backslash M(K_4)$, $P_9 \cong \mathbb{P}\backslash[M(k_4-e)\oplus U_{1,1}]$, $M^*(K_{3,3}) \cong \mathbb{P}\backslash[U_{2,3}\oplus U_{2,3}]$ and $M(K_5\backslash e) \cong \mathbb{P}\backslash P(U_{2,3}, U_{3,4})$, with 9 elements;*
(iii) *$\mathbb{P}\backslash M(K_4\backslash e)$, $\mathbb{P}\backslash[U_{2,3} \oplus U_{2,2}]$, $\mathbb{P}\backslash[U_{3,4} \oplus U_{1,1}]$ and $M(K_5) \cong \mathbb{P}\backslash U_{4,5}$, with 10 elements;*
(iv) *$\mathbb{P}\backslash[U_{2,3} \oplus U_{1,1}]$, $\mathbb{P}\backslash U_{3,4}$ and $\mathbb{P}\backslash U_{4,4}$; with 11 elements; and:*
(v) *$\mathbb{P}\backslash U_{1,1}$, $\mathbb{P}\backslash U_{2,2}$, $\mathbb{P}\backslash U_{2,3}$ and $\mathbb{P}\backslash U_{3,3}$ with more than 11 elements.*

Proof. In this proof when we're talking about equality and uniqueness, it is up to isomorphisms. We're denoting by M an arbitrary 3-connected rank-4 binary matroid.

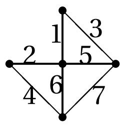
By the main result of Oxley [14], the only 3-connected rank-4 binary matroids with no $M(\mathcal{W}_4)$-minor and are F_7^*, $AG(3,2)$, S_8, and Z_4. So, the rank-4 binary 3-connected matroids up to 8 elements are F_7^*, $AG(3,2)$, S_8 and $M(\mathcal{W}_4)$. Let us find the complementaries of these matroids in relation to $\mathbb{P}$.

It is easy to see that $AG(3,2) = \mathbb{P}\backslash F_7$. As the unique single-element deletion of F_7 is $M(K_4)$, it follows that the unique rank-4 simple binary single-element extension of $AG(3,2)$ is $\mathbb{P}\backslash M(K_4) =$

Z_4. So, the complementary of $S_8 = Z_4\backslash b_4$ is the unique single-element extension of $M(K_4)$ different from F_7, that is $M(K_4) \oplus U_{1,1}$. We also have that $M(\mathscr{W}_4) = \mathbb{P}\backslash M(\mathscr{W}_4\backslash b)$, were b is an edge in the rim of $\mathscr{W}_4$, as shown in the graphs and the respective matrices that represent then below.



$$\begin{array}{c} \begin{matrix}1&2&3&4&5&6&7&8\end{matrix} \\ \begin{pmatrix} 0&1&1&1&1&0&0&1\\ 1&0&1&1&1&1&0&0\\ 1&1&0&1&0&1&1&0\\ 1&1&1&0&0&0&1&1 \end{pmatrix}\end{array}$$

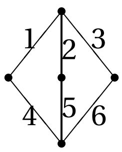


$$\begin{array}{c} \begin{matrix}1&2&3&4&5&6&7\end{matrix} \\ \begin{pmatrix} 1&0&0&0&1&0&1\\ 0&1&0&0&0&1&1\\ 0&0&1&0&1&0&1\\ 0&0&0&1&0&1&1 \end{pmatrix}\end{array}$$

As the only excluded minors for graphicness in binary matroids are F_7, F_7^*, $M^*(K_5)$ and $M^*(K_{3,3})$, then all binary matroids up to 6 elements are graphic. So, if $|E(M)| \geq 9$, then $\mathbb{P}\backslash M$ is graphic.

The only possible degree sequences for simple connected graphs with 6 edges and 4 or 5 vertices are: $(3,3,3,3)$, $(2,2,2,2,4)$, $(2,2,2,3,3)$ and $(1,2,2,3,4)$. Indeed, if 4 appear twice in such a sequence, then the graph has at least 7 edges. So 4 appears at most once. The sum of the degrees must be 12. So, as 4 appears at most once, 1 appears at most once too. Now it is easy to check that the possible sequence are those listed above. This implies that the unique simple matroids with 6 elements and rank up to 4 are: $M(K_4)$, $P(U_{2,3}, U_{2,3})$, $M(K_{2,3})$, $U_{2,3} \oplus U_{3,4}$ and $M(K_4 - e) \oplus U_{1,1}$.

Below we can see, in this order, a draw of $K_{3,3}$ and matrices representing $M(K_{2,3})$ and $\mathbb{P}\backslash M(K_{2,3})$:



$$\begin{array}{c} \begin{matrix}1&2&3&4&5&6\end{matrix} \\ \begin{pmatrix} 0&1&1&1&0&0\\ 1&0&1&0&1&0\\ 1&1&0&0&0&1\\ 1&1&1&1&1&1 \end{pmatrix}\end{array} \qquad \begin{array}{c} \begin{matrix}1&2&3&4&5&6&7&8&9\end{matrix} \\ \begin{pmatrix} 1&0&0&0&1&1&0&1&1\\ 0&1&0&0&1&0&1&1&1\\ 0&0&1&0&0&1&1&1&1\\ 0&0&0&1&0&0&0&0&1 \end{pmatrix}\end{array}$$

Note that the last row in the second matrix corresponds to a 2-cocircuit of $\mathbb{P}\backslash M(K_{2,3})$, which is not 3-connected therefore. Note that all proper restrictions of $M(K_{2,3})$ are restrictions of $M(K_4) \oplus U_{1,1}$. So all 3-connected rank-4 binary matroids with at least 9 elements are complementaries of some restriction of $M(K_4) \oplus U_{1,1}$ or $M(\mathscr{W}_4\backslash b)$. In other hand, if M is a restriction of $M(K_4) \oplus U_{1,1}$ or $M(\mathscr{W}_4\backslash b)$, then $\mathbb{P}\backslash M$ is a rank-4 extension of S_8 or $M(\mathscr{W}_4)$, and, therefore, M is 3-connected. This description corresponds to the matroids listed in the theorem. $\square$

8. Computational results

In this section we describe briefly the methods and procedures used to prove Lemma 4.4. To get the list of non-separating cocircuits of a binary matroid M and count how many of then avoid each element, we just use a brutal force algorithm that examines each linear combination of lines in a standard matrix representing M. The subroutines for this are based on well known algorithms.

Consider a binary matrix A with columns $c_1, \dots, c_n$, and $v = (v_1, \dots, v_n) \in \{0,1,2\}^n$. Let, for each, $i = 1, \dots, n$, $\bar{v_i}$ be the reminder of the division of v_i by two. Moreover, let $i_1 < \dots < i_k$ be the elements of $\{j : 1 \leq j \leq n \text{ and } v_j = 2\}$. We define

$$\Gamma(A, v) := \left(\begin{array}{c|c|c|c|c|c|c} 1 & \bar{v_1} & \dots & \bar{v_n} & 1 & \dots & 1 \\ \hline 0 & c_1 & \dots & c_n & c_{i_1} & \dots & c_{i_k} \end{array}\right).$$

It is easy to check that, for binary matroids M and N and a binary matrix A, if $M \cong M[A]$ and there is $e \in E(N)$ such that $M \cong si(N/e)$, then there is $v \in \{0,1,2\}^n$ such that $N \cong M[\Gamma(A, v)]$. The definition of Γ may look awkward at a first moment, but it is easy to deal computationally, and make it easier to enumerate such matroids N as above. Another attractive property of Γ is equation (5) below.

For a binary matrix A with columns labelled by $1,\dots,n$ and for an automorphism σ of $M[A]$ it is straight to check that:

$$M[\Gamma(A,(v_1,\dots,v_n))]\cong M[\Gamma(A,(v_{\sigma(1)},\dots,v_{\sigma(n)}))]. \tag{5}$$

The next lemma is a straight consequence of Bixby's Theorem about decomposition of non 3-connected matroids into 2-sums.

Lemma 8.1. *Let N be a coloopless simple non-3-connected binary matroid with at least 4 elements. Suppose that e is an element of N such that $si(N/e)$ is 3-connected. Then e belongs to a non-trivial series class of N.*

From this lemma we conclude:

Corolary 8.2. *Let A be a binary matrix with $n\ge 4$ columns and $v\in\{0,1,2\}^n$. If $M[A]$ is 3-connected and $M[\Gamma(A,v)]$ is cossimple, then $M[\Gamma(A,v)]$ is 3-connected.*

Let $A:=[I_r|D]$ be an $r\times n$ binary matrix. We define: $\mathscr{L}(A):=\{\Gamma(A,v);v\in\{0,2\}^r\times\{0,1,2\}^{n-r}\}$. Let $\mathscr{M}$ be a family of binary matroids we define a family of binary matrices $\mathscr{A}$ to be a **standard vector representation** of $\mathscr{M}$ if each matroid of $\mathscr{M}$ is isomorphic to $M[A]$ for some $A\in\mathscr{A}$ and all matrices in $\mathscr{A}$ are in the standard form. For a family of binary matrices $\mathscr{A}$ we denote $M[\mathscr{A}]:=\{M[A]:A\in\mathscr{A}\}$ and $M^*[\mathscr{A}]:=\{M^*[A]:A\in\mathscr{A}\}$. We simplify the language saying that standard vector representation of $M[A]$ is a **standard vector representation** of $\mathscr{A}$.

Lemma 8.3. *If A is a binary matrix and M is a matroid with an element e such that $M[A]\cong co(M\backslash e)$ and $|E(M)|-|E(A(M))|\le k$, then M is isomorphic to a matroid represented by a matrix in $\mathscr{L}^k(A)$.*

The following lemma describe the procedures that are being used to prove Conjecture 1.5.

Lemma 8.4. *If M is a 3-connected regular matroid, with rank 5, with a $M(K'''_{3,3})$-minor and with no $M(K_5)$-minor, then $M\cong M(K'''_{3,3})$.*

Proof. Since, M has no R_{12} minor and $M\ncong R_{10}$ and M is not cographic, them M is graphic. So, if $M\ncong K'''_{3,3}$, then M is the cycle matroid of a 6-vertex simple graph properly extending $K'''_{3,3}$. The result follows from Lemma 4.2. □

Lemma 8.5. *Suppose that M' is a 3-connected regular matroid with no $M^*(K_5)$ and with an $M^*(K'''_{3,3})$-minor such that $r^*(M')\le 9$ and $r^*_{M'}(\widetilde{Y}(M'))\ge 3$. Then a matroid isomorphic to M' can be found with the following procedure:*

(1) Let A_0 be a standard binary matrix representing $M(K'''_{3,3})$. Let $\mathscr{A}_6$ be a standard vector representation of
$\{A\in\mathscr{L}A_0,\ M^*[A]$ is regular, 3-connected and has no K_5-minor and $\widetilde{Y}(M^*[A])\neq\emptyset\}$.
Check if all the matroids $M\in M^*[\mathscr{A}_6]$ satisfy $\widetilde{Y}(M^*)\le 2$.
(2) Let $\mathscr{L}_6=\mathscr{A}_6$. For $i=7,8,9$ do the following step;
(3) Let
$$\mathscr{L}_i:=\bigcup_{A\in\mathscr{L}_{i-1}}\mathscr{L}(A)$$
and let $\mathscr{A}_i$ be a set of representatives of the following family:
$\{A\in\mathscr{L}_i, M[A]\ M$ is regular, 3-connected and has no K_5-minor and $r^*\widetilde{Y}(M^*[A])\ge 2\}$.
Check if all the matroids $M\in M^*[\mathscr{A}_i]$ satisfy $\widetilde{Y}(M^*)\le 2$.

Proof of Lemma 8.5 : We have to prove that if such M' exists, then M'^* is isomorphic to a matroid in $M^*[\mathscr{A}_6]\cup\cdots\cup M^*[\mathscr{A}_9]$. Suppose for a contradiction that this does not hold.

Let $r = r^*(M')$. By Lemma 2.1 and by Lemma 8.4, there is a chain of matroids $M(K'''_{3,3}) = M_5, M_6, \dots M_r = M'^*$, such that for each $i = 6, \dots, r$ there is an element $e_i \in E(M_i)$ such that $M_{i-1} = si(M_i/e_i)$.

If $r = 6$ it is clear that there is a matroid in $M^*[\mathscr{A}_6]$ isomorphic do M. So $r \geq 7$.

If $r = 7$, since $r(M') - r(M(K'''_{3,3})) \geq 2$, by Lemma 3.7 for $l = 0$, $r_{M_6}(\widetilde{Y}(M_6^*)) \geq 1$. So M'^* is isomorphic to a matroid in $\mathscr{A}_7$. Hence $r \geq 8$.

If $r = 8$, since $r(M') - r(M(K'''_{3,3})) \geq 3$, by Lemma 3.7, for $l = 1$, $r_{M_7}(\widetilde{Y}(M_7^*)) \geq 2$. So M'^* is isomorphic to a matroid in $\mathscr{A}_8$. Hence $r = 9$. Analogously we prove that $M' \in \mathscr{A}_9$ and arrive at a contradiction. □

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